INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 5, 981–993 (1985)

FINITE ELEMENT ANALYSIS OF DENSITY FLOW USING THE VELOCITY CORRECTION METHOD

MUTSUTO KAWAHARA AND KIYOTAKA OHMIYA

Department of Civil Engineering, Chuo University Kasuga, Bunkyo-ku, Tokyo, Japan

SUMMARY

A finite element method is proposed for the analysis of density flow which is induced by a difference of density. The method employs the idea that density variation can be pursued by using markers distributed in the flow field. For the numerical integration scheme, the velocity correction method is successfully used, introducing a potential for the correction of velocity. This method is useful because one can use linear interpolation functions for velocity, pressure and potential based on the triangular finite element. The final equations can be formulated using the quasi-explicit finite element method. A flume in a tank with sloping bottom has been analysed by the present method. The computed results show extremely good agreement with the experimental observations.

KEY WORDS Finite Element Method Velocity Correlation Method Density Flow Density Flume Linear Interpolation

INTRODUCTION

There are a number of engineering problems which require one to solve the problem of a density flow induced by the difference of fluid density. In recent years, much research concerned with such density flows has been presented, based on numerical analysis. One of the common analysis methods is referred to as the MAC method;[1–5] this uses many markers distributed in the flow field and moves them to pursue the loci of fluid particles. These analyses are mainly performed by the finite difference method. The basic idea can be successfully extended to the finite element analysis. This paper presents the finite element method for a density flow in which the density variation is accounted for by markers distributed in the flow field and moved according to the fluid velocity.

The authors' research group has previously presented the finite element method based on the two step explicit numerical integration scheme.[6–8] An advantageous point is that computational core storage can be saved by the explicit computation. This is because the computation can be carried out mainly by the multiplication of coefficients and velocity and pressure vectors. Contrary to this, it takes a comparatively long computational time because the time increment must be taken short enough. Recent development of computer hardware makes it possible that a simultaneous equation system with a large number of unknowns can be solved easily. Thus, this paper presents the finite element method based on the quasi-explicit numerical integration scheme. After lengthy and careful investigations, it was decided to use the velocity correction method among the various quasi-explicit schemes.

The basic idea of the velocity correction method was originally presented by Chorin.[9] Donea *et al.*,[10,11] Schneider and Raithby,[12,13] Glowinsky *et al.*[14] and Mizukami and Tsuchiya[15] adapted the idea to the finite element method. If one knows the velocity and pressure at an instant, the

0271–2091/85/110981–13$01.30

Received 20 June 1984
Revised 20 June 1985

velocity after a short time increment can be obtained using the discretized equation of motion. However, the computed velocity would not in general satisfy the equation of continuity. Thus, the computed velocity must be corrected to satisfy the equation of motion. In this procedure, the correction velocity is assumed to be derived from a potential. Then, the simultaneous equations can be formulated with the potentials chosen to be the unknown variables. The solution of the simultaneous equations leads to the correction velocity and pressure. The whole procedure can be repeated until a previously assigned stage is reached.

To illustrate the validity of the present method, a density flume sliding down the inclined bottom of a tank filled with fluid is analysed. The computed flume behaviour is quite well in agreement with experimental observations. The velocity correction method has been shown to be a useful tool for the analysis of density flow.

BASIC EQUATIONS

The analysis presented in this paper is concerned with the mixing phenomena of two different fluids. The density distribution is not uniform in the entire flow field. However, the incompressibility condition and Boussinesq approximation are assumed valid because the time variation of density is not significant. The density distribution is computed from the manner in which the fluid particles are traced by markers attached to the fluid. The transient viscous incompressible flow can be expressed by the Navier–Stokes equations. All equations are written using the indicial notation and the usual summation convention for repeated indices. Denoting time by t, co-ordinates by $x_i (i = 1, 2)$, velocity by u_i and pressure by p, the equations of motion and continuity can be described in the following forms:

$$\dot{u}_i + u_j u_{i,j} + \frac{1}{\rho_0} p_{,i} - \nu(u_{i,j} + u_{j,i})_{,j} - \frac{\rho}{\rho_0} f_i = 0, \tag{1}$$

$$u_{i,i} = 0, \tag{2}$$

where the superposed dot and subscripted comma mean differentiation with respect to time t and co-ordinate x_i, respectively, ρ and ρ_0 represent density and reference density and ν and f_i are viscosity and body force, respectively.

As boundary conditions, the conditions for velocity u_i and surface force t_i are introduced as follows:

$$u_i = \hat{u}_i, \qquad \text{on } S_1, \tag{3}$$

$$t_i = \left\{ -\frac{1}{\rho_0} p\delta_{ij} + \nu(u_{i,j} + u_{j,i}) \right\} n_j = \hat{t}_i, \qquad \text{on } S_2, \tag{4}$$

where the superposed $\hat{}$ denotes the functions which are given on the boundary, n_j means the unit normal to the boundary and S_1 and S_2 are boundaries on which u_i and t_i are prescribed, respectively. The boundary S of the flow field V consists of S_1 and S_2 only, and S_1 and S_2 do not overlap. There is a free surface on a part of the boundary S_2 at which the position of the surface moves according to the fluid movement. The position of the surface is governed by the equation

$$\frac{\partial \eta}{\partial t} + u_i^{(s)} \eta_{,i} = 0, \tag{5}$$

where η is the surface elevation measured from the fundamental fluid level and $u_i^{(s)}$ is the velocity

at the surface. As the initial condition, density, velocity and pressure are given as function of the spatial co-ordinate:

$$\rho = \rho^{(0)}(x_i), \tag{6}$$

$$u_i = u_i^{(0)}(x_i), \tag{7}$$

$$p = p^{(0)}(x_i), \tag{8}$$

where $\rho^{(0)}$, $u_i^{(0)}$, $p^{(0)}$ are density, velocity and pressure at time $t = 0$, respectively.

VELOCITY CORRECTION METHOD

For the finite element method for equations (1)–(8), the velocity correction method is usefully applied in this paper. The algorithm is derived in this section. The total time to be analysed is divided into a large number of short time intervals, one of which is expressed by Δt. An arbitrary time point which separates time intervals is denoted by n. A superscript n means that the function is evaluated at time point n. Equation (1) can be approximated as follows:

$$\bar{u}_i^{n+1} = u_i^n - \Delta t\left\{u_j^n u_{i,j}^n + \frac{1}{\rho_0} p_{,i}^n - \nu(u_{i,j}^n + u_{j,i}^n)_{,j} - \frac{\rho}{\rho_0} f_i\right\}, \tag{9}$$

where $\bar{u}_i^{n+1}$ means the approximated velocity computed by equation (9). The velocity $\bar{u}_i^{n+1}$ can be computed at time point $(n+1)$ using the values of u_i^n and p^n. But this computed velocity would not satisfy the equation of continuity (2). Denoting the exact velocity by u_i^{n+1} and the pressure by p^{n+1} at time point $(n+1)$, the following equation is valid:

$$u_i^{n+1} = u_i^n - \Delta t\left\{u_j^n u_{i,j}^n + \frac{1}{\rho_0} p_{,i}^{n+1} - \nu(u_{i,j}^n + u_{j,i}^n)_{,j} - \frac{\rho}{\rho_0} f_i\right\}, \tag{10}$$

where p^{n+1} is defined as the pressure obtained by equation (10). The velocity u_i^{n+1} can be derived by correcting the approximate velocity $\bar{u}_i^{n+1}$ as

$$u_i^{n+1} = \bar{u}_i^{n+1} + u_i', \tag{11}$$

where u_i' means the correction velocity. Looking at equations (9)–(11), the correlation velocity can be derived from a potential ϕ as follows:

$$u_i' = \phi_{,i}. \tag{12}$$

The velocity u_i^{n+1} must satisfy equation (2). Thus, introduction of equations (11) and (12) into equation (2) leads to

$$u_{i,i}^{n+1} = \bar{u}_{i,i}^{n+1} + \phi_{,ii} = 0. \tag{13}$$

From equation (13), it follows that

$$\phi_{,ii} = -\bar{u}_{i,i}^{n+1}. \tag{14}$$

Introducing the boundary conditions for the potential ϕ as

$$\phi = \hat{\phi}, \qquad \text{on } S_2, \tag{15}$$

$$r = \phi_{,i} n_i = \hat{r}, \qquad \text{on } S_1, \tag{16}$$

equation (14) can be solved to determine the potential ϕ. Substituting equations (9) and (10) into

equation (11) and rearranging the terms, the pressure equation can be derived in the form

$$p_{,i}^{n+1} = p_{,i}^{n} + \frac{\rho_0}{\Delta t}\phi_{,i}. \tag{17}$$

Therefore, the pressure at time point $(n+1)$ can be computed in the following form:

$$p^{n+1} = p^{n} - \frac{\rho_0}{\Delta t}\phi, \tag{18}$$

where the constant of integration is assumed to be zero.

FINITE ELEMENT METHOD

Based on the velocity v_i^n and pressure p^n computed at the previous time point, the velocity v_i^{n+1} and pressure p^{n+1} can be computed using equations (9), (11), (12) and (14). Those equations are discretized by the finite element method. Assume that the flow field is divided into a number of small domains called finite elements. For velocity and pressure, the interpolation equations can be introduced as follows:

$$u_i = \Phi_\alpha u_{\alpha i}, \tag{19}$$

$$p = \Phi_\alpha p_\alpha, \tag{20}$$

where Φ_α is the interpolation function, $u_{\alpha i}$ denotes the nodal value of velocity at the αth node of the finite element in the ith direction, and p_α means the nodal value of pressure at the αth node. The application of the finite element procedure to equation (9) leads to the following equation:

$$\bar{M}_{\alpha\beta}\bar{u}_{\beta i}^{n+1} = \tilde{M}_{\alpha\beta}u_{\beta i}^{n} - \Delta t(K_{\alpha\beta\gamma j}u_{\beta j}^{n}u_{\gamma i}^{n} - \frac{1}{\rho_0}H_{\alpha i\beta}p_\beta^n + S_{\alpha i\beta j}u_{\beta j}^{n} - \frac{\rho}{\rho_0}F_{\alpha i} + \hat{\Omega}_{\alpha i}), \tag{21}$$

where

$$M_{\alpha\beta} = \int_V (\Phi_\alpha \Phi_\beta)\,\mathrm{d}V,$$

$$K_{\alpha\beta\gamma j} = \int_V (\Phi_\alpha \Phi_{\beta,j} \Phi_\gamma)\,\mathrm{d}V,$$

$$H_{\alpha i\beta} = \int_V (\Phi_{\alpha,i} \Phi_\beta)\,\mathrm{d}V,$$

$$S_{\alpha i\beta j} = \int_V \nu(\Phi_{\alpha,i}\Phi_{\beta,j})\,\mathrm{d}V + \int_V \nu(\Phi_{\alpha,k}\Phi_{\beta,k})\delta_{ij}\,\mathrm{d}V,$$

$$F_{\alpha i} = \int_V (\Phi_\alpha f_i)\,\mathrm{d}V,$$

$$\hat{\Omega}_{\alpha i} = \int_{S_2} (\Phi_\alpha \hat{f}_i)\,\mathrm{d}S.$$

In equation (21), $\bar{M}_{\alpha\beta}$ is the lumped coefficient of $M_{\alpha\beta}$ and $\tilde{M}_{\alpha\beta}$ is the mixed coefficient, so that

$$\tilde{M}_{\alpha\beta} = e\bar{M}_{\alpha\beta} + (1-e)M_{\alpha\beta}, \tag{22}$$

where e is the lumping parameter. Applying the finite element procedure to equations (11), (12), and (14), the following equations can be derived:

$$\bar{M}_{\alpha\beta}u_{\beta i}^{n+1} = \bar{M}_{\alpha\beta}u_{\beta i}^{n} + H_{\beta i\alpha}\phi_{\beta}, \tag{23}$$

$$A_{\alpha\beta}\phi_{\beta} = -H_{\beta i\alpha}\bar{u}_{\beta i}^{n+1} + \hat{G}_{\alpha}, \tag{24}$$

where

$$A_{\alpha\beta} = \int_{V} (\Phi_{\alpha,i}\Phi_{\beta,i})\,\mathrm{d}V,$$

$$\hat{G}_{\alpha} = \int_{S_1} (\Phi_{\alpha}\hat{r})\,\mathrm{d}S.$$

In the derivation of equations (23) and (24), the potential is interpolated using the following equation:

$$\phi = \Phi_{\alpha}\phi_{\alpha} \tag{25}$$

where Φ_{α} is the interpolation equation, which is the same as in equations (19) and (20), and ϕ_{α} represents the nodal value of the potential at the αth node of the finite element.

COMPUTATIONAL PROCEDURES

In this paper, two phase flow in which the densities are ρ_1 and ρ_2 is analysed. A number of markers are distributed in the flow field to distinguish fluid properties. Two kinds of markers are used corresponding to the fluids of densities ρ_1 and ρ_2. The position of a marker is moved according to

$$x_i^{n+1} = x_i^n + \Delta t u_i^n, \tag{26}$$

where x_i^n is the position of the marker at time point n. The density of fluid is computed at each finite element. Let the numbers of markers at each finite element of the fluids with densities ρ_1 and ρ_2 be m_1 and m_2, respectively. The density of a finite element (i) can be computed as follows:

$$\rho^{(i)} = \frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2}, \tag{27}$$

where $\rho^{(i)}$ means the density of the finite element (i). The reference density is computed by the weighted average of finite elements at the initial stage:

$$\rho_0 = \frac{\sum_{i=1}^{M} \rho^{(i)}A^{(i)}}{\sum_{i=1}^{M} A^{(i)}} \tag{28}$$

where $A^{(i)}$ is an area of finite element (i) and M is the total number of finite elements. The algorithm of the velocity correction method is summarized as follows:

Step 1. Set u_i^n, p^n and x_i^n
Step 2. Computation of $\rho^{(i)}$ by equation (27)
Step 3. Computation of $\bar{u}_i^{n+1}$ by equation (21)
Step 4. Computation of ϕ by equation (24)
Step 5. Computation of u_i^{n+1} by equation (23)

Step 6. Computation of p^{n+1} by equation (18)
Step 7. Movement of markers by equation (26)
Step 8. Replace u_i^n, p^n and x_i^n by u_i^{n+1}, p^{n+1} and x_i^{n+1} and repeat from Step 2.

The algorithm is referred to as the quasi-explicit finite element method. This is due to the fact that the simultaneous equation (24) must be solved, whereas the explicit computation can be applied to equations (21), (23) and (18) because the lumped coefficients are introduced on the left side of these equations.

ANALYSIS OF FLUME IN TANK WITH SLOPE

A schematic view of the tank used for the analysis is illustrated in Figure 1. The boundary A–B–C–D–E is the solid wall, on which the following conditions are assumed:

$$\left.\begin{aligned} u_n = u_i n_i = 0 \\ r = \phi_{,i} n_i = 0 \end{aligned}\right\} \quad \text{on A–B–C–D–E,} \tag{29, 30}$$

where u_n is the velocity normal to the boundary. Equation (29) means the so-called slip condition. This condition is employed because the viscosity is assumed to be zero in the present condition. The boundary A–F–E is the free surface on which the following assumptions are introduced:

$$\left.\begin{aligned} t_i = \left\{ -\frac{1}{\rho_0} p\delta_{ij} + \nu(u_{i,j} + u_{j,i}) \right\} n_j = 0 \\ \phi = 0 \end{aligned}\right\} \quad \text{on A–F–E.} \tag{31, 32}$$

From equations (18) and (32), the pressure is expressed as

$$p^{n+1} = p^n, \quad \text{on A–F–E,} \tag{33}$$

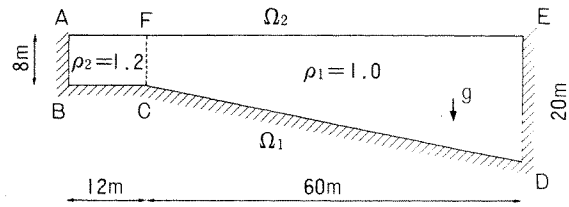

Figure 1. Schematic view of the tank

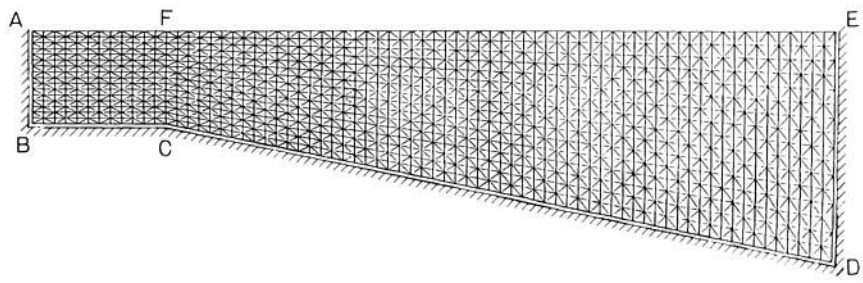

Figure 2. Finite element idealization

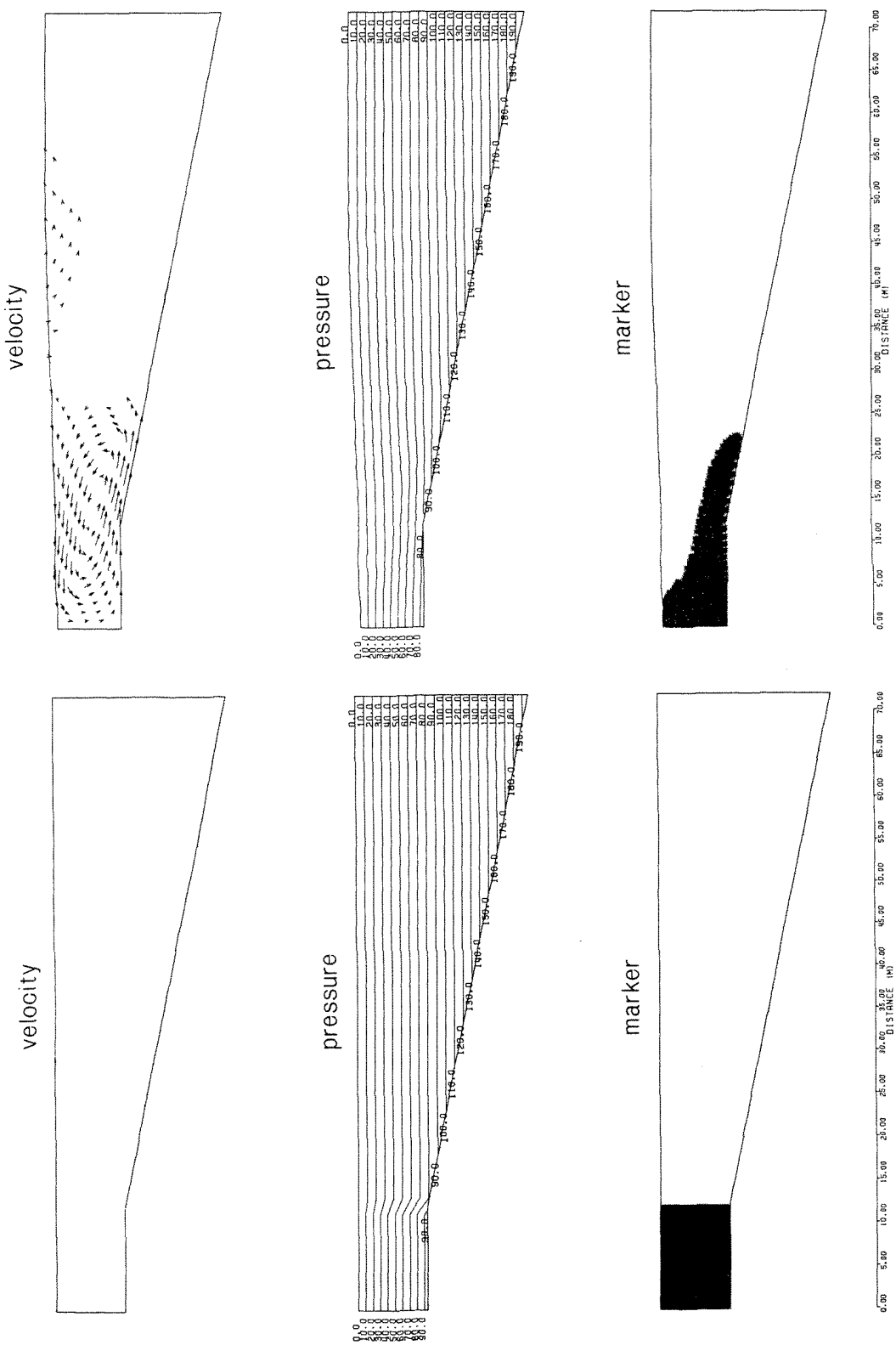

Figure 4. Computed results at step 60($T = 6$)

Figure 3. Initial conditions

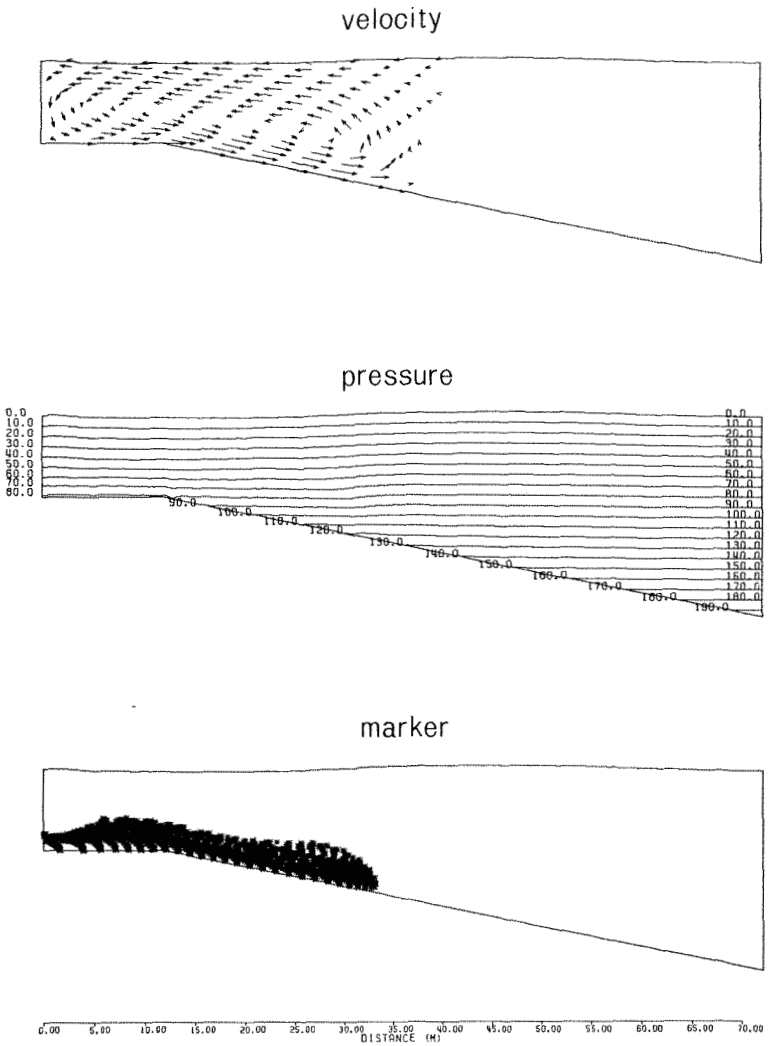

Figure 5. Computed results at step 120($T=12$)

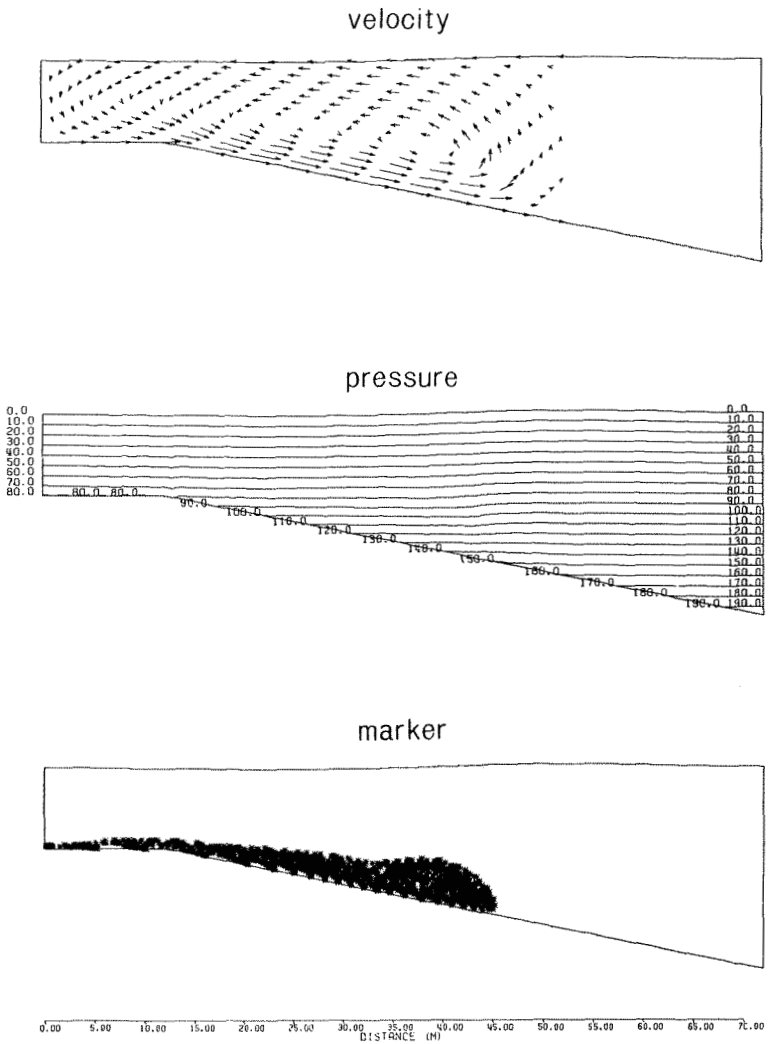

Figure 6. Computed results at step 180($T=18$)

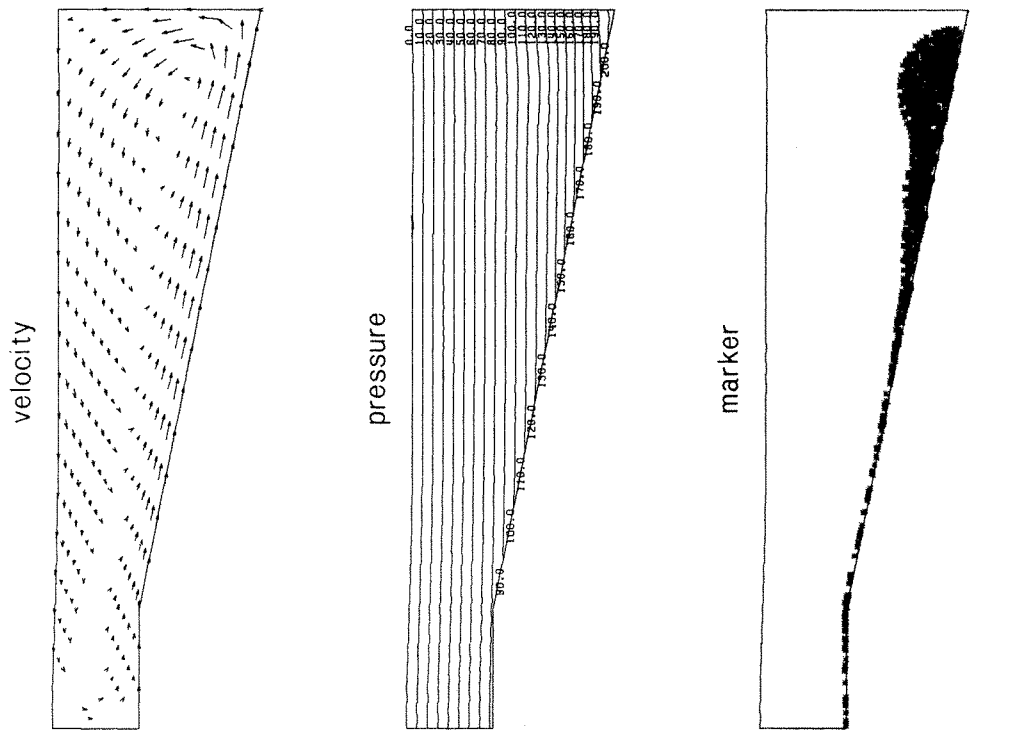

Figure 8. Computed results at step 300($T = 30$)

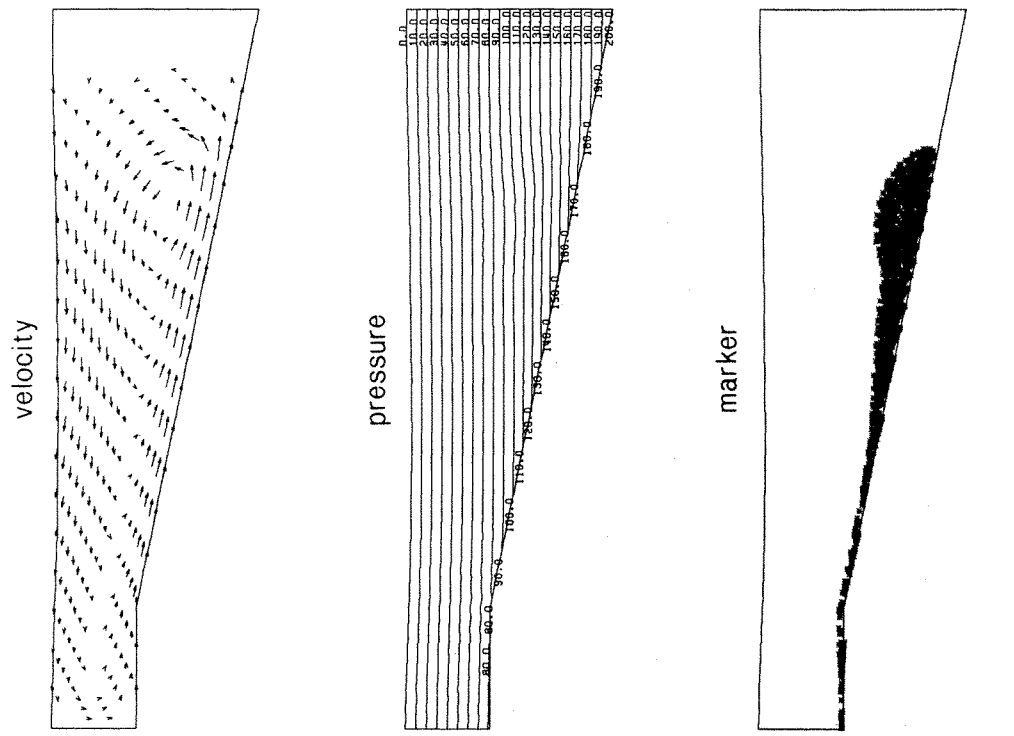

Figure 7. Computed results at step 240($T = 24$)

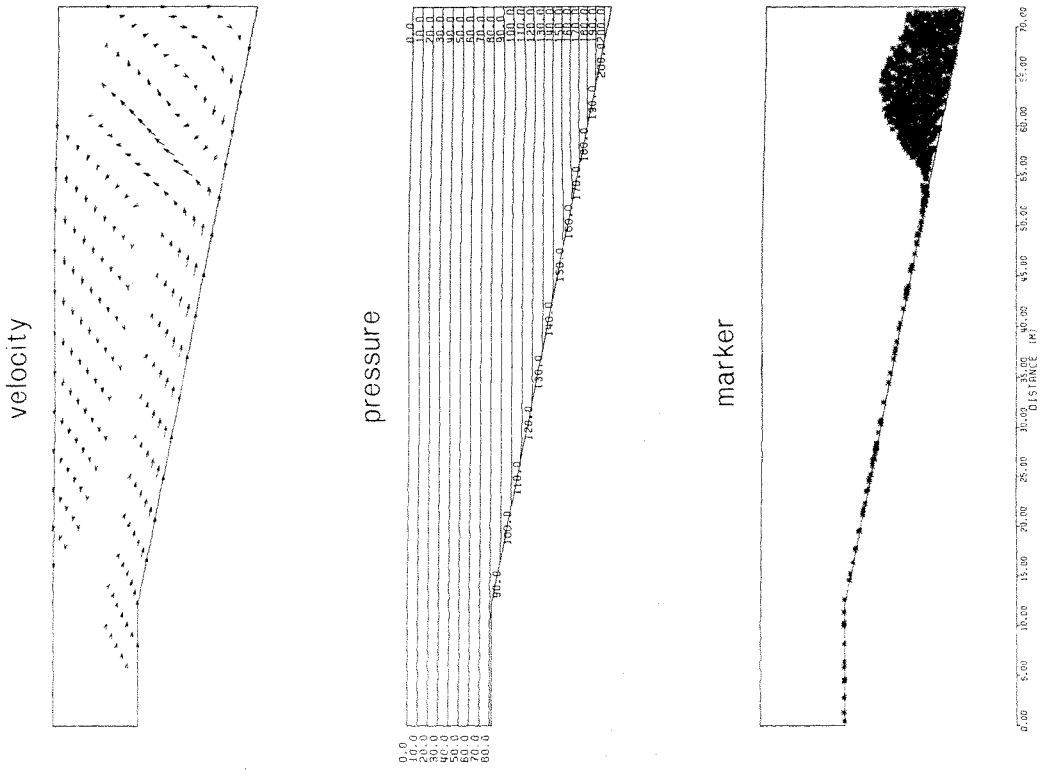

Figure 10. Computed results at step 420($T = 42$)

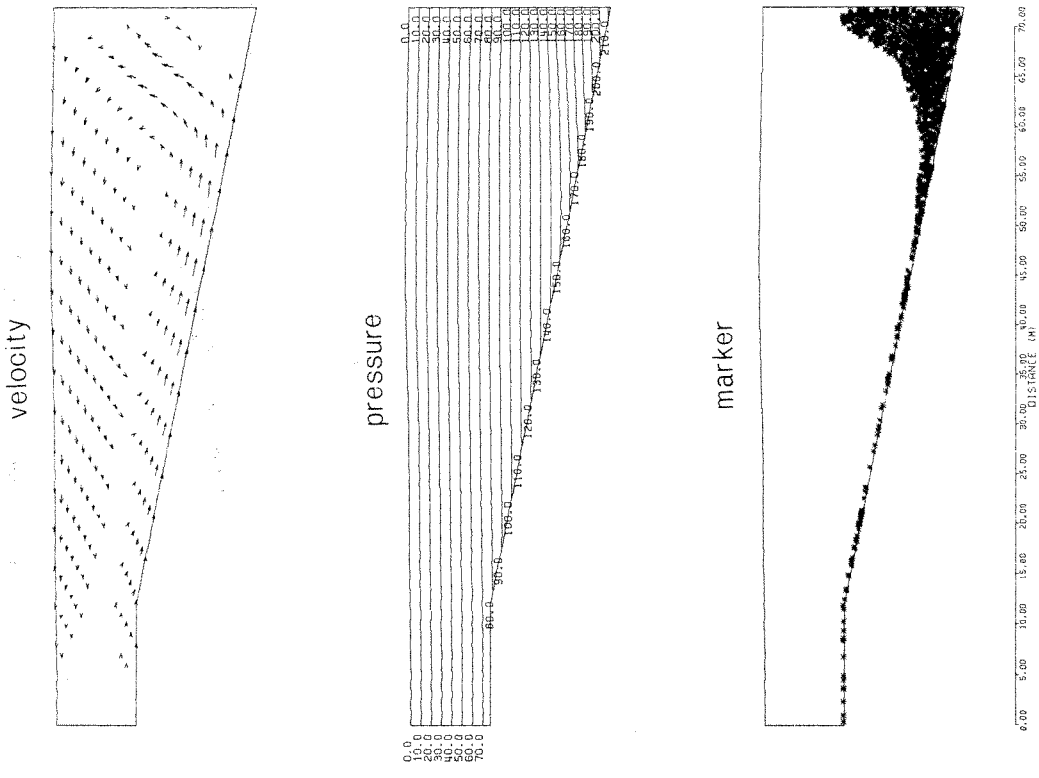

Figure 9. Computed results at step 360($T = 36$)

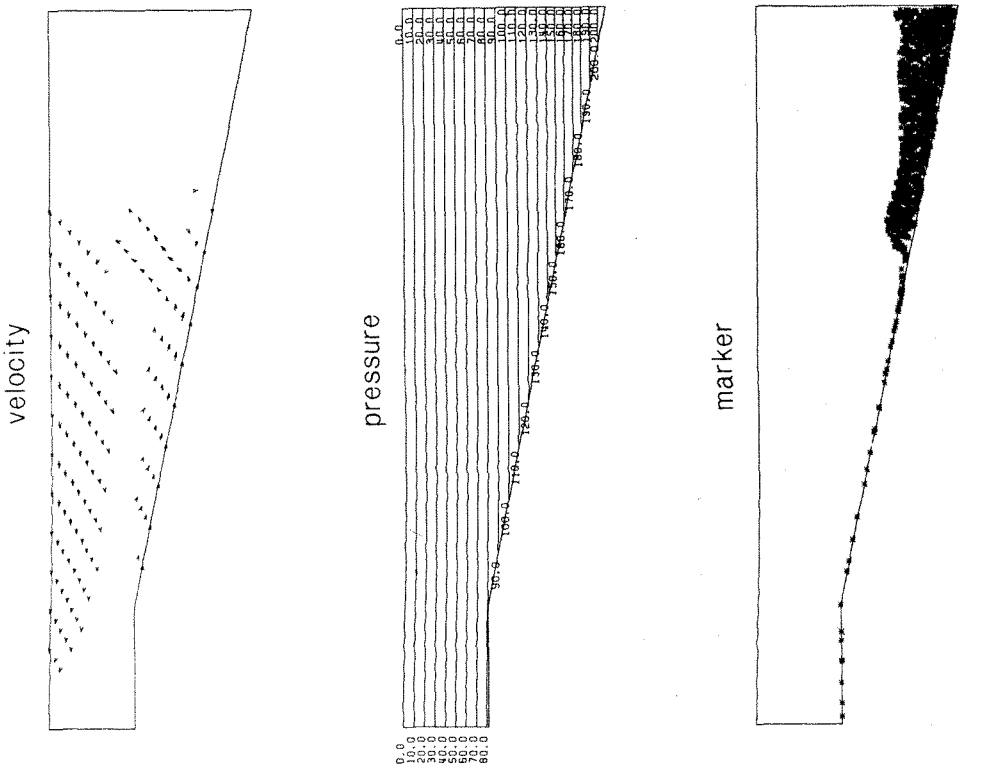

Figure 12. Computed results at step 540($T = 54$)

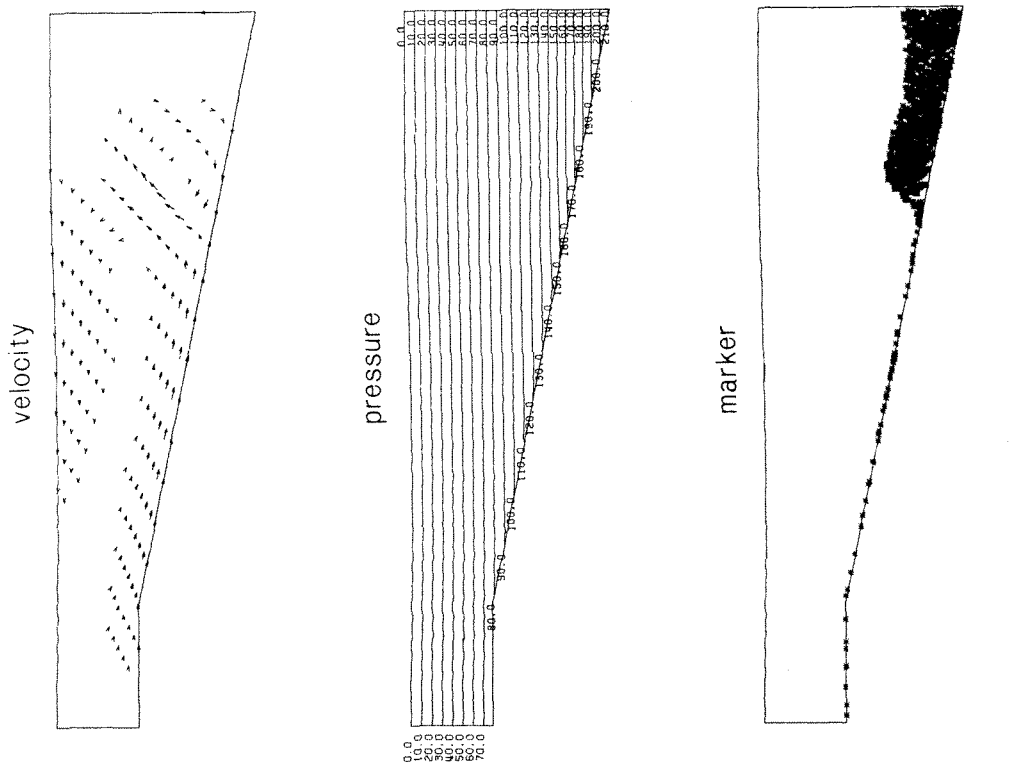

Figure 11. Computed results at step 480($T = 48$)

which means that the pressure at the free surface is always the reference pressure. The surface elevation of the free surface is determined by

$$\frac{\partial \eta}{\partial t} + u_i^{(s)} \eta_{,i} = 0, \quad \text{on A–F–E,} \tag{34}$$

which can be discretized based on the linear finite element approximation.

For the interpolation equations of velocity, pressure and potential, the linear interpolation function based on triangular finite elements is used. The finite element idealization is shown in Figure 2. The total numbers of nodal points and finite elements are 1241 and 1984, respectively. The initial conditions of velocity, pressure and markers are represented in Figure 3. The velocity distribution is assumed to be zero everywhere in the tank and the pressure is hydrostatic. The domain A–B–C–F is filled with fluid with density $\rho_2 = 1{\cdot}2$ and F–C–D–F with fluid with density $\rho_1 = 1{\cdot}0$. The gravity acceleration $g = 9{\cdot}8$ and viscosity $\nu = 0{\cdot}0$ are used. The time interval $\Delta t = 0{\cdot}1$ time units is employed with the lumping parameter $e = 0{\cdot}5$. The computed velocity, pressure and markers are shown in Figures 4–12. For clarity, the markers of the fluid ρ_1 are all omitted. The computation started assuming that the barrier C–F was suddenly removed. Figures 4–12 are illustrations of the flume sliding down along the bottom of the tank at every 60 time units. Figure 4 shows the high density fluid starting to move along the slope of the bottom. The movement behaviour of the density wedge can be clearly seen in Figures 5–8. Especially, the shape of the wedge front computed is in very good agreement with the experimental observations. Figure 9 represents the configuration at the instant when the wedge front arrives at any runs up to the solid wall. Figures 10–12 clearly show the behaviour in which the high density fluid sinks down dynamically under the low density fluid. The configuration of the free surface is also in good agreement with the appearance observed in the experiments.

ACKNOWLEDGEMENTS

The authors are grateful to Dr. T. Nakayama, K. Kashiyama and B. Ramaswamy, Department of Civil Engineering, Chuo University for their earnest and helpful discussions during the preparation of this paper. Computations have been carried out using FACOM M170F of Chuo University and HITAC S810 of the University of Tokyo.

REFERENCES

1. J. E. Welch, F. H. Harlow, J. P. Shannon and B. J. Daly, 'The MAC method: a computing technique for solving viscous, incompressible, transient fluid flow problems involving free surface', *Los Alamos Scientific Laboratory Report, LA-3425*, 1966.
2. F. H. Harlow and J. E. Welch, 'Numerical calculation of time dependent viscous incompressible flow of fluid with free surface', *Phys. Fluid.*, **8**, 2182–2189 (1965).
3. B. J. Daly and W. E. Pracht, 'Numerical study of density current surge', *Phys. Fluid.*, **11**, 15–30 (1968).
4. A. A. Amsden and F. H. Harlow, 'The SMAC method: a numerical technique for calculating incompressible fluid flows', *Los Alamos Scientific Laboratory Report, LA-4370*, 1970.
5. C. W. Hirt, A. A. Amsden and J. L. Cook, 'An arbitrary Lagrangian–Eulerian computing method for all flow speeds', *J. Comp. Phys.*, **14**, 227–253 (1974).
6. M. Kawahara, H. Hirano, K. Tsubota and K. Inagaki, 'Selective lumping finite element method for shallow water flow', *Int. j. numer. methods fluids*, **2**, 89–112 (1982).
7. M. Kawahara and H. Hirano, 'A Finite element method for high Reynolds number viscous fluid flow using two step scheme', *Int. j. numer. methods fluids*, **3**, 137–163 (1983).
8. M. Kawahara and H. Hirano, 'Two step explicit finite element method for high Reynolds number viscous fluid flow', *Proc. Japan Soc. Civil Eng.*, No. 329, 127–140 (1983).
9. A. J. Chorin, 'Numerical solution of the Navier–Stokes equations', *Math. Comp.*, **22**, 745–762 (1968).

10. J. Donea, S. Giuliani, H. Laval and L. Quartapelle: 'Solution of the unsteady Navier–Stokes equations by a finite element projection method', in C. Taylor and K. Morgan (eds), *Computational Techniques in Transient and Turbulent Flow*, Pineridge Press, 1981, pp. 97–132.
11. J. Donea, S. Giuliani and H. Laval, 'Finite element solution of the unsteady Navier–Stokes equations by a fractional step method', *Comp. Meth. Appl. Mech. Engng.*, **32**, 53–73 (1982).
12. G. E. Schneider, G. D. Raithby and M. M. Yovanovich, 'Finite element solution procedures for solving the incompressible, Navier–Stokes equations using equal order variable interpolation', *Numerical Heat Transfer*, **1**, 435–451 (1978).
13. G. E. Schneider and G. D. Raithby, 'Finite element analysis of incompressible fluid flow incorporating equal order pressure and velocity interpolation', in K. Morgan *et al.* (eds) *Computer Methods in Fluids*, Pentech Press, 1980, pp. 49–83.
14. R. Glowinsky, B. Mantel, J. Periaux, P. Perrier and O. Pirronneau 'On an efficient new preconditioned conjugate gradient method. Application to the in-core solution of the Navier–Stokes equations via non-linear least-square and finite element method', in R. H. Gallagher *et al.* (eds), *Finite Elements in Fluids*, Vol. 4, Wiley, Chichester, 1982, pp. 365–401.
15. A. Mizukami and M. Tsuchiya, 'A finite element method for the three dimensional non-steady Navier–Stokes equations', *Int. j. numer. methods fluids*, **4**, 349–357 (1984).